JOURNAL OF APPROXIMATION THEORY **45**, 167–193 (1985)

On the Approximation of Holomorphic Functions by Müntz Polynomials on an Interval away from the Origin

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Communicated by G. Meinardus

Received October 23, 1984

The problem of the approximation of functions by Müntz polynomials on an interval $[a, b]$ with $a>0$ is considered. It is shown that, in contrast to the behaviour in approximating on an interval $[0, b]$, for the approximation of a function f by Müntz polynomials on $[a, b]$, $a>0$, the minimal deviation tends to zero with a geometric rate for all functions f holomorphic in a sufficiently large region around the interval $[a, b]$. Also a converse theorem is given. The results are interpreted in the context of the equivalent (linear) problem of the approximation of functions by exponential sums. © 1985 Academic Press, Inc.

1. INTRODUCTION

Let $C[a, b]$ denote the space of real valued continuous functions on $[a, b]$, $a<b$, endowed with the uniform norm

$$\|f\|_{[a,b]} := \max\{|f(x)|: x\in[a, b]\}, \qquad f\in C[a, b].$$

We consider the approximation of functions $f\in C[a, b]$, $a>0$, by Müntz polynomials from $\Pi_n(\lambda_\nu)$,

$$\Pi_n(\lambda_\nu) := \left\{\sum_{\nu=0}^{n} a_\nu x^{\lambda_\nu}: a_\nu \in \mathbb{R}\right\}, \qquad n\in\mathbb{N}, \tag{1}$$

on the interval $[a, b]$, where (λ_ν) denotes a fixed sequence of real numbers λ_ν, $\nu\in\mathbb{N}$:

$$0\leqslant \lambda_0<\lambda_1<\cdots, \qquad \lim_{\nu\to\infty}\lambda_\nu=\infty. \tag{2}$$

The classical Müntz theorem (cf. [12]) states that the system of polynomials from $\Pi_n(\lambda_\nu)$ is dense in $C[0, 1]$ iff $\sum_{\nu=1}^{\infty}(1/\lambda_\nu)=\infty$ and

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$\lambda_0=0$. The corresponding result for the approximation on an interval $[a, b]$ with $a>0$ has been proved independently by Clarkson and Erdös (cf. [4]) and L. Schwartz (cf. [16]). Given $f\in C[a, b]$ we are interested in the asymptotic behaviour of the minimal deviation $\rho_n(f, (\lambda_\nu), [a, b])$,

$$\rho_n(f, (\lambda_\nu), [a, b]) := \min\{\|f-p_n\|_{[a,b]} : p_n \in \Pi_n(\lambda_\nu)\} \tag{3}$$

as $n\to\infty$.

Looking at the transformation

$$\begin{aligned} x&=e^{-t}, & t&\in[-\log b, -\log a], & F(t)&=f(e^{-t}),\\ t&=-\log x, & x&\in[a, b], & f(x)&=F(-\log x), \end{aligned} \tag{4}$$

we see that the problem of the approximation of a function $f(x)\in C[a, b]$, $0\leqslant a<b<\infty$, by Müntz polynomials from $\Pi_n(\lambda_\nu)$ on $[a\, b]$ is equivalent to the problem of the approximation of $F(t)\in C[\alpha, \beta]$ with $\alpha=-\log b$, $\beta=-\log a$ on $[\alpha, \beta]$ by help of exponential sums from $\Delta_n(\lambda_\nu)$,

$$\Delta_n(\lambda_\nu) := \left\{\sum_{\nu=0}^{n} a_\nu e^{-\lambda_\nu t},\ a_\nu\in\mathbb{R}\right\}, \qquad n\in\mathbb{N}. \tag{5}$$

The corresponding minimal deviation reads

$$\delta_n(F, (\lambda_\nu), [\alpha, \beta]) := \min\{\|F-d_n\|_{[\alpha,\beta]} : d_n\in\Delta_n(\lambda_\nu)\}. \tag{6}$$

Mostly these equivalent problems are treated as Müntz approximation problems although only the approximation by exponential sums seems to be of practical relevance.

For the rate of convergence of the minimal deviation $\rho_n(f, (\lambda_\nu), [a, b])$, $f\in C[a, b]$, as $n\to\infty$ Jackson-type theorems have been proved by several authors. (See, e.g., M. v. Golitschek [8], D. Leviatan [10], D. J. Newman [13] for the approximation on $[0, b]$, and M. v. Golitschek [9] for the approximation on $[a, b]$, $a>0$.) In contrast to the usual approximation by algebraic polynomials for the Müntz approximation on the interval $[0, b]$ the order of approximation need not generally increase with the smoothness of the function being approximated. For example, let us regard the approximation of $f(x)=x$ by polynomials $x^{d\nu}$, $\nu\in\mathbb{N}$, $d>0$, $1/d\notin\mathbb{N}$ (i.e., $\lambda_\nu=d\nu$, $\nu\in\mathbb{N}$) on $[0, 1]$. Replacing x by $x^{1/d}$ it follows

$$\rho_n(x, (\nu d), [0, 1]) = \rho_n(x^{1/d}, (\nu), [0, 1]), \qquad n\in\mathbb{N}.$$

The minimal deviation in approximating $g(x)=x^{1/d}$ by usual polynomials on $[0, 1]$ is of order $n^{-2/d}$ (cf. G. Meinardus [11]). Thus we have the asymptotic relation

$$\rho_n(x, (\nu d), [0, 1]) = O(n^{-2/d}) \qquad \text{as } n\to\infty. \tag{7}$$

But for the Müntz approximation on intervals $[a, b]$ with $a > 0$ the rate of $\rho_n(f, (\lambda_\nu), [a, b])$ depends on the smoothness of f just as in the case of usual approximation by polynomials (cf. M. v. Golitschek [9]).

We are now interested in the question of what kind of functions $f \in C[a, b]$ can be approximated by Müntz polynomials from $\Pi_n(\lambda_\nu)$ with a geometric rate of convergence, i.e.,

$$\rho_n(f, (\lambda_\nu), [a, b]) \leqslant A \cdot q^{-n}, \qquad n \in \mathbb{N}, \text{ with } q > 1.$$

For the approximation on the interval $[0, 1]$ this problem has been regarded in [14]. The results suggest that under the assumption $0 < d \leqslant \lambda_{\nu+1} - \lambda_\nu$, $\nu \in \mathbb{N}$, on the sequence (λ_ν) the minimal deviation $\rho_n(f, (\lambda_\nu), [0, 1])$ tends to zero geometrically if and only if the approximated function f is the restriction of a "Müntz series"

$$\hat{f}(z) = \sum_{\nu=0}^{\infty} c_\nu z^{\lambda_\nu}, \qquad c_\nu \in \mathbb{R},\ z \in \mathbb{C}_{\log}, \tag{8}$$

absolutely convergent in a certain domain around the branch point zero of the Riemann surface of the logarithm. (Since the numbers λ_ν are allowed to be irrational the complex value z in (8) must be an element from the Riemann surface of the logarithm denoted by $\mathbb{C}_{\log}$.)

Again the situation is changed for the approximation on intervals $[a, b]$ with $a > 0$. Let us consider the approximation of $f(x) = x$ by polynomials $x^{d\nu}$, $\nu \in \mathbb{N}$, $d > 0$, $1/d \notin \mathbb{N}$, on the interval $[a, 1]$. By substituting x for $x^{1/d}$ we get

$$\left\| x - \sum_{\nu=0}^{n} a_\nu x^{d\nu} \right\|_{[a,1]} = \left\| x^{1/d} - \sum_{\nu=0}^{n} a_\nu x^{\nu} \right\|_{[a^d,1]}$$

and consequently

$$\rho_n(x, (\nu d), [a, 1]) = \rho_n(x^{1/d}, (\nu), [a^d, 1]), \qquad n \in \mathbb{N}.$$

The function $g(x) = x^{1/d}$ is holomorphic in any circle not containing the zero point. Thus by the theorem of S. N. Bernstein (cf. G. Meinardus [11]) we find by taking account of the transformation of the interval $[-1, 1]$ onto $[a, 1]$ that the geometric decrease

$$\rho_n(x, (\nu d), [a, 1]) = O\left(\frac{1 + \sqrt{a^d}}{1 - \sqrt{a^d}}\right)^{-n}, \qquad n \in \mathbb{N},$$

occurs in contrast to the approximation on $[0, 1]$ (cf. (7)).

In the following we are mainly interested in the behaviour of the minimal deviation $\rho_n(f, (\lambda_\nu), [a, b])$ in approximating a function f by Müntz polynomials on intervals $[a, b]$, $a > 0$, where the function f is assumed to be holomorphic around the approximation interval $[a, b]$.

2. A Special Müntz Approximation Problem

M. Hasson has regarded (cf. [5, 6]) the special "Müntz problem" of the approximation of functions by usual polynomials p_n in which for a fixed $k \in \mathbb{N}$ the coefficients of x^k are zero, i.e.,

$$p_n(x) = \sum_{\substack{\nu=0 \\ \nu \neq k}}^{n} a_\nu x^\nu, \qquad n \in \mathbb{N}.$$

We define for $k, n \in \mathbb{N}$, $k \leqslant n$, $f \in C[a, b]$ the minimal deviation

$$\rho_n^{(k)}(f, (\nu), [a, b]) := \min_{a_\nu} \left\{ \left\| f(x) - \sum_{\substack{\nu=0 \\ \nu \neq k}}^{n} a_\nu x^\nu \right\|_{[a,b]} \right\}. \tag{9}$$

First we look at the approximation of the function $f(x) = x^k$ by the other power functions x^ν, $\nu \in \mathbb{N}$, $\nu \neq k$. This example reflects some peculiarities of the general Müntz approximation problem.

The alternating properties of the Chebyshev polynomials of the first kind T_n, $n \in \mathbb{N}$, lead to the following

Lemma 1. *Let a, b, $0 \leqslant a < b$, be given. Then relation*

$$\rho_n^{(k)}(x^k, (\nu), [a, b]) = \frac{k!\,(b-a)^k}{2^k \left| T_n^{(k)}\left(1 + \dfrac{2a}{b-a}\right) \right|} \tag{10}$$

holds for all $k, n \in \mathbb{N}$, $k \leqslant n$.

Proof. Considering the polynomials $T_n(2(x-a)/(b-a) - 1)$ on $[a, b]$ the assertion follows by the same arguments as for the special case $a = 0$, $b = 1$ in M. Hasson [5].

Since (cf. A. F. Timan [15, p. 226])

$$T_n^{(k)}(1) = \prod_{\nu=1}^{k} \frac{n^2 - (\nu-1)^2}{2\nu - 1} = n^{2k} \frac{2^k k!}{(2k)!} \prod_{\nu=1}^{k} 1 - \left(\frac{\nu-1}{n}\right)^2, \quad n \geqslant k \geqslant 1,$$

we get by (10)

$$\rho_n^{(k)}(x^k, (\nu), [0, b]) = \prod_{\nu=1}^{k} \frac{1}{1 - \left(\dfrac{\nu-1}{n}\right)^2} \cdot \frac{b^k (2k)!}{4^k} n^{-2k}$$

for $1 \leqslant k \leqslant n$. Using

$$\left(\frac{2k-1}{k^2}\right)^k = \prod_{\nu=1}^{k} 1 - \left(\frac{k-1}{k}\right)^2 \leqslant \prod_{\nu=1}^{k} 1 - \left(\frac{\nu-1}{n}\right)^2 < 1, \qquad n \geqslant k \geqslant 1,$$

we find inequality

$$\frac{b^k (2k)!}{4^k} n^{-2k} < \rho_n^{(k)}(x^k, (\nu), [0, b]) \leqslant \left(\frac{k^2}{2k-1}\right)^k \frac{b^k(2k)!}{4^k} n^{-2k}, \qquad (11)$$

$n \geqslant k \geqslant 1$, for the minimal deviation $\rho_n^{(k)}(x^k, (\nu), [0, b])$ in approximating x^k by the other power functions x^ν, $\nu \in \mathbb{N}$, $\nu \neq k$, on the interval $[0, b]$ (cf. also M. Hasson [5]). But moving the approximation interval away from the origin the minimal deviation $\rho_n^{(k)}(x^k, (\nu), [a, b])$, $a > 0$, tends to zero geometrically.

LEMMA 2. *Let $a, b, 0 < a < b$, and $k \in \mathbb{N}$ be given. Then the asymptotic relation*

$$\rho_n^{(k)}(x^k, (\nu), [a, b]) = 2k!(\sqrt{ab})^k n^{-k} \left(\frac{\sqrt{b}+\sqrt{a}}{\sqrt{b}-\sqrt{a}}\right)^{-n} \left(1 + O\left(\frac{1}{n}\right)\right) \qquad (12)$$

holds as $n \to \infty$.

Proof. With transformation

$$z = \tfrac{1}{2}(v + 1/v), \qquad v = z + \sqrt{z^2 - 1},\ z = x + iy,\ v \in \mathbb{C},$$

equality

$$T_n(z) = \tfrac{1}{2}(v^n + 1/v^n)$$

is valid for the Chebyshev polynomials T_n, $n \in \mathbb{N}$. This follows by applying the identity principle for holomorphic functions to relation

$$\tfrac{1}{2}(e^{in\varphi} + e^{-in\varphi}) = \cos n\varphi = T_n(\cos \varphi),$$

$v = e^{i\varphi}$, $z = \cos \varphi$, $\varphi \in \mathbb{R}$. Setting $D_x := d/dx$ we obtain with

$$T_n(x) = \tfrac{1}{2}(v^n + 1/v^n), \qquad z = x > 1,\ v = x + \sqrt{x^2 - 1}, \qquad (13)$$

by differentiation

$$D_x T_n(x) = \frac{d}{dv}\left(\frac{1}{2}\left(v^n + \frac{1}{v^n}\right)\right) \cdot D_x v = \frac{1}{2}\left(n v^{n-1} - \frac{n}{v^{n+1}}\right) \cdot \frac{x + \sqrt{x^2-1}}{\sqrt{x^2-1}}$$
$$= \frac{1}{2} n \left(v^n - \frac{1}{v^n}\right) \frac{1}{\sqrt{x^2-1}}$$

and further for $k \leqslant n$

$$\begin{aligned} D_x^k T_n(x) &= \frac{1}{2} n^k \left(v^n + (-1)^k \frac{1}{v^n} \right) \left(\frac{1}{\sqrt{x^2-1}} \right)^k \\ &\quad + \sum_{\nu=1}^{k-1} \frac{1}{2} n^\nu \left(v^n + (-1)^\nu \frac{1}{v^n} \right) F_\nu(x) \end{aligned} \tag{14}$$

with functions F_ν, $\nu = 1(1)\,k-1$,

$$F_\nu(x) = F_\nu \left(\frac{1}{\sqrt{x^2-1}}, D_x \frac{1}{\sqrt{x^2-1}}, ..., D_x^{k-1} \frac{1}{\sqrt{x^2-1}} \right),$$

not depending on $n \in \mathbb{N}$. Hence combining (13) and (14) we find setting $\kappa := 1 + 2a/(b-a) > 1$, $r := \kappa + \sqrt{\kappa^2 - 1}$ for any fixed $k \in \mathbb{N}$ the asymptotic relation

$$T_n^{(k)}(\kappa) = \frac{1}{2} \left(\frac{1}{\sqrt{\kappa^2-1}} \right)^k n^k r^n \left(1 + O\left(\frac{1}{n}\right) \right) \qquad \text{as } n \to \infty.$$

With (10) it follows that

$$\rho_n^{(k)}(x^k, (\nu), [a, b]) = \frac{k!\,(b-a)^k (\sqrt{\kappa^2-1})^k}{2^{k-1}} n^{-k} r^{-n} \left(1 + O\left(\frac{1}{n}\right) \right)$$

as $n \to \infty$. In view of

$$\sqrt{\kappa^2 - 1} = \frac{2}{b-a} \sqrt{ba}, \qquad r = \kappa + \sqrt{\kappa^2-1} = \frac{\sqrt{b} + \sqrt{a}}{\sqrt{b} - \sqrt{a}}$$

we have established relation (12).

Remark 1. Since

$$\frac{\sqrt{b} + \sqrt{a}}{\sqrt{b} - \sqrt{a}} = \frac{(\sqrt{b} + \sqrt{a})^2}{b-a}$$

we see from (12) that for fixed length $b-a$ of the interval $[a, b]$ the geometric rate of the minimal deviation $\rho_n^{(k)}(x^k, (\nu), [a, b])$ increases with the distance of the interval from the zero point.

A geometric rate of the minimal deviation $\rho_n^{(k)}(f, (\nu), [a, b])$, $a > 0$, also occurs for all functions f holomorphic around the approximation interval $[a, b]$. This is stated in

THEOREM 1. *Let* $0<a<b$ *and* $k \in \mathbb{N}$ *be given. Let* $E((b-a)/2+\kappa)$ *denote the ellipse with foci* a, b *and great half axis of length* $(b-a)/2+\kappa$,

$$E\left(\frac{b-a}{2}+\kappa\right)=\left\{z=x+iy: x=\left(\frac{b-a}{2}+\kappa\right)\cos\psi+\frac{b+a}{2},\right. \tag{15}$$
$$\left. y=\sqrt{\left(\frac{b-a}{2}+\kappa\right)^2-1}\,\sin\psi,\ \psi\in[0,2\pi)\right\}$$

where κ *is a number* $0<\kappa\leqslant a$. *Suppose the function* f *is holomorphic in a region containing the ellipse* $E((b-a)/2+\kappa)$, $0<\kappa\leqslant a$, *and its interior. Further let* $f(z)$ *be real for real* z. *Then inequality*

$$\rho_n^{(k)}(f,(v),[a,b])\leqslant A\left(\frac{(\sqrt{b-a+\kappa}+\sqrt{\kappa})^2}{b-a}\right)^n, \qquad n\in\mathbb{N}, \tag{16}$$

holds with a constant A *not depending on* n.

Proof. Let

$$f(x)=\sum_{v=0}^{\infty} a_v T_v\left(\frac{2(x-a)}{b-a}-1\right) \tag{17}$$

be the Chebyshev expansion of f on the interval $[a,b]$. By assumption f is holomorphic on an ellipse $E((b-a)/2+\kappa_1)$ (cf. (15)) and its interior with $\kappa_1=\kappa+\varepsilon$, $\varepsilon>0$, small enough. By applying transformation

$$\tilde{x}=\frac{2(x-a)}{b-a}-1, \qquad x\in[a,b],\ \tilde{x}\in[-1,1],$$

mapping the interval $[a,b]$ onto $[-1,1]$ and the point $a-\kappa_1$ on $-(1+2\kappa_1/(b-a))$ we obtain just as in the proof of the Bernstein theorem (cf. G. Meinardus [11, p. 91]) the bound

$$|a_v|\leqslant\frac{2B(\kappa_1)}{R_1^v}, \qquad v\in\mathbb{N}, \text{ where } \qquad \begin{aligned} q_1 &:= 1+\frac{2\kappa_1}{b-a},\\ R_1 &:= q_1+\sqrt{q_1^2-1},\\ B(\kappa_1) &:= \max_{z\in E((b-a)/2+\kappa_1)}|f(z)|. \end{aligned} \tag{18}$$

The inequality of S. N. Bernstein (cf. [11, p. 92]) yields

$$\left|T_v\left(\frac{2(z-a)}{b-a}-1\right)\right|\leqslant R_2^v, \qquad v\in\mathbb{N},\ z\in E\left(\frac{b-a}{2}+\kappa_2\right),$$

for any $\kappa_2 > 0$ with $R_2 = q_2 + \sqrt{q_2^2 - 1}$, $q_2 = 1 + 2\kappa_2/(b-a)$. Now we choose a number κ_2, $\kappa_1 > \kappa_2 > \kappa$. Since $R_1 > R_2$ it follows by (18) that

$$\begin{aligned} |f(z)| &\leqslant \sum_{v=0}^{\infty} |a_v| \left| T_v\left(\frac{2(z-a)}{b-a} - 1\right)\right| \\ &< 2B(\kappa_1) \sum_{v=0}^{\infty} \left(\frac{R_2}{R_1}\right)^v < \infty \end{aligned} \qquad \text{for } z \in E\left(\frac{b-a}{2} + \kappa_2\right) \tag{19}$$

and the Chebyshev expansion (17) is uniformly convergent on the ellipse $E((b-a)/2 + \kappa_2)$ and its interior. Hence

$$\lim_{n \to \infty} \|f - p_n\|_{E((b-a)/2 + \kappa_2)} = 0 \tag{20}$$

with the partial sums

$$p_n(z) = \sum_{v=0}^{n} a_v T_v\left(\frac{2(z-a)}{b-a} - 1\right) = \sum_{v=0}^{n} \alpha_v^{(n)} z^v, \qquad n \in \mathbb{N}, \tag{21}$$

of (17) and consequently

$$\begin{aligned} \|p_n\|_{E((b-a)/2+\kappa_2)} &\leqslant \|f - p_n\|_{E((b-a)/2+\kappa_2)} + \|f\|_{E((b-a)/2+\kappa_2)} \\ &\leqslant A_1 \|f\|_{E((b-a)/2+\kappa_2)} \end{aligned} \tag{22}$$

with a constant $A_1 > 0$.

Let us first assume $a > \kappa_2 > \kappa$. The transformation

$$\tilde{z} = \frac{2(z-a)}{b-a} - 1$$

maps

$$a - \kappa_2 \text{ resp. } 0 \text{ on } -\left(1 + \frac{2\kappa_2}{b-a}\right) \text{ resp. } -\left(1 + \frac{2a}{b-a}\right)$$

and thus the ellipses

$$E\left(\frac{b-a}{2} + \kappa_2\right) \text{ resp. } E\left(\frac{b+a}{2}\right) \text{ onto } \tilde{E}\left(1 + \frac{2\kappa_2}{b-a}\right) \text{ resp. } \tilde{E}\left(1 + \frac{2a}{b-a}\right).$$

Here $\tilde{E}(\sigma)$, $1 < \sigma$, denotes the ellipse with foci $-1, 1$ and great half axis of length σ. Setting

$$q_0 := 1 + \frac{2a}{b-a}, \qquad q_2 := 1 + \frac{2\kappa_2}{b-a}$$

we see that transformation

$$\tilde{z} = \tfrac{1}{2}(v + 1/v) \tag{23}$$

maps

$$\begin{aligned} &\{v \in \mathbb{C} \colon |v| = 1\} && \text{onto } [-1, 1], \\ &\{v \in \mathbb{C} \colon |v| = R_0 := q_0 + \sqrt{q_0^2 - 1}\} && \text{onto } \tilde{E}\left(1 + \frac{2a}{b-a}\right), \\ &\left\{v \in \mathbb{C} \colon |v| = R_2 := q_2 + \sqrt{q_2^2 - 1}\right\} && \text{onto } \tilde{E}\left(1 + \frac{2\kappa_2}{b-a}\right). \end{aligned} \tag{24}$$

Moreover by (23) we have given a conformal mapping between the region $\{v \in \mathbb{C} \colon |v| > R_2\}$ and the exterior $\operatorname{ext} \tilde{E}(1 + 2\kappa_2/(b-a))$ of the ellipse $\tilde{E}(1 + 2\kappa_2/(b-a))$. Hence setting $\tilde{v} := v/R_2$ by

$$\tilde{z} = \frac{1}{2}\left(R_2 \tilde{v} + \frac{1}{R_2 \tilde{v}}\right), \qquad \tilde{v} = \frac{1}{R_2}(\tilde{z} + \sqrt{\tilde{z}^2 - 1})$$

there is given a mapping between

$$\begin{aligned} &\{\tilde{v} \in \mathbb{C} \colon |\tilde{v}| = 1\} && \text{and} && \tilde{E}\left(1 + \frac{2\kappa_2}{b-a}\right) \\ &\left\{\tilde{v} \in \mathbb{C} \colon |\tilde{v}| = \frac{R_0}{R_2}\right\} && \text{and} && \tilde{E}\left(1 + \frac{2a}{b-a}\right) \end{aligned}$$

and a conformal mapping between

$$\{\tilde{v} \in \mathbb{C} \colon |\tilde{v}| > 1\} \qquad \text{and} \qquad \operatorname{ext} \tilde{E}\left(1 + \frac{2\kappa_2}{b-a}\right).$$

We remember that $a > \kappa_2$ or $R_0 > R_2$. By a generalisation of the inequality of S. N. Bernstein (cf. D. Gaier [7, S. 33]) we obtain with (22), taking account of the above transformations, that

$$\begin{aligned} |p_n(z)| &\leqslant \|p_n\|_{E((b-a)/2 + \kappa_2)} \left(\frac{R_0}{R_2}\right)^n, \\ &< A_2 \left(\frac{R_0}{R_2}\right)^n, \end{aligned} \qquad n \in \mathbb{N}, \tag{25}$$

is valid for all $z \in E((b+a)/2)$. With aid of the inequality of W. A. Markoff (cf. G. Meinardus [11]) applied to the interval $[0, b+a]$ comprised in the ellipse $E((b+a)/2)$ it follows

$$k!\,|\alpha_k^{(n)}| = |p_n^{(k)}(0)| \leqslant A_2 \left(\frac{R_0}{R_2}\right)^n \frac{2^{2k}k!}{(b+a)^k} \frac{n}{n+k} \binom{n+k}{n-k},$$

and thus

$$|\alpha_k^{(n)}| \leqslant A_3\, n^{2k} \left(\frac{R_0}{R_2}\right)^n, \qquad n \in \mathbb{N}, \tag{26}$$

for the kth coefficients $\alpha_k^{(n)}$ of p_n (cf. (21)), $k \leqslant n$, k fixed, with a constant A_3 not depending on n.

Now let $q_n \in \Pi_n \backslash \operatorname{span}(x^k)$ be the polynomials which best approximate the function x^k on $[a, b]$, i.e.,

$$\rho_n^{(k)}(x^k, (v), [a, b]) = \|x^k - q_n\|_{[a,b]}, \qquad n \in \mathbb{N}.$$

With the partial sums p_n, $n \in \mathbb{N}$ (cf. (21)), we have

$$\begin{aligned} \rho_n^{(k)}(f, (v), [a, b]) &\leqslant \|f - p_n + \alpha_k^{(n)} x^k - \alpha_k^{(n)} q_n\|_{[a,b]} \\ &\leqslant \|f - p_n\|_{[a,b]} + |\alpha_k^{(n)}|\, \rho_n^{(k)}(x^k, (v), [a, b]). \end{aligned} \tag{27}$$

From (18) and (21) we see that for all $x \in [a, b]$ the bound

$$\begin{aligned} |f(x) - p_n(x)| &\leqslant \sum_{v=n+1}^{\infty} |a_v| \left| T_v\left(\frac{2(x-a)}{b-a} - 1\right)\right| \\ &\leqslant 2B(\kappa_1) \sum_{v=n+1}^{\infty} R_1^{-v}, \qquad n \in \mathbb{N}, \\ &\leqslant C_1 R_1^{-n} \end{aligned} \tag{28}$$

is correct. Now with $q_0 = 1 + 2a/(b-a)$

$$R_0 = q_0 + \sqrt{q_0^2 - 1} = \frac{\sqrt{b} + \sqrt{a}}{\sqrt{b} - \sqrt{a}}$$

Lemma 1 yields for fixed $k \in \mathbb{N}$

$$\rho_n^{(k)}(x^k, (v), [a, b]) \leqslant C_2\, n^{-k} R_0^{-n}, \qquad n \in \mathbb{N}. \tag{29}$$

Using (26) we get

$$|\alpha_k^{(n)}|\, \rho_n^{(k)}(x^k, (v), [a, b]) \leqslant C_3\, n^k R_2^{-n}, \qquad n \in \mathbb{N}. \tag{30}$$

In view of $\kappa_1 > \kappa_2 > \kappa$ or $R_1 > R_2$ we deduce from (27), (28), and (30) that

$$\rho_n^{(k)}(f, (v), [a, b]) \leqslant B n^k R_2^{-n}, \qquad n \in \mathbb{N}. \tag{31}$$

Setting $q := 1 + 2\kappa/(b-a)$, $R := q + \sqrt{q^2 - 1}$ it follows by $\kappa_2 > \kappa$ that $R_2 > R$ and thus

$$\lim_{n \to \infty} n^k \left(\frac{R}{R_2}\right)^n = 0.$$

This combined with (31) yields for fixed $k \in \mathbb{N}$ inequality

$$\rho_n^{(k)}(f, (v), [a, b]) \leqslant A R^{-n}, \qquad n \in \mathbb{N},$$

where the constant A is not depending on n. Since

$$R = \frac{(\sqrt{b - a + \kappa} + \sqrt{\kappa})^2}{b - a}$$

the assertion of the theorem is proved for $\kappa < a$.

Now let us take $\kappa = a$. As before we obtain with suitable $\kappa_1, \kappa_2; \kappa_1 > \kappa_2 > a$, inequality (19) and thus

$$\lim_{n \to \infty} \|f - p_n\|_{E((b-a)/2 + \kappa_2)} = 0 \tag{32}$$

with the partial sums p_n, $n \in \mathbb{N}$ (cf. (21)),

$$p_n(z) = \sum_{v=0}^{n} \alpha_v^{(n)} z^v.$$

The circle $\mathring{K}_\sigma = \{z \in \mathbb{C} : |z| < \sigma\}$ with $\sigma := \kappa_2 - a$ is contained in the interior of the ellipse $E((b-a)/2 + \kappa_2)$ and by Cauchy's integral formula we find with the power series

$$f(z) = \sum_{v=0}^{\infty} c_v z^v$$

of f relation

$$f^{(k)}(0) - p_n^{(k)}(0) = \frac{k!}{2\pi i} \int_{|\xi| = \sigma} \frac{f(\xi) - p_n(\xi)}{\xi^{k+1}} \, d\xi$$

and

$$|c_k - \alpha_k^{(n)}| = \frac{f^{(k)}(0) - p_n^{(k)}(0)}{k!} \leqslant \frac{\|f - p_n\|_{E((n-a)/2 + \kappa_2)}}{\sigma^k}. \tag{33}$$

Hence by (32) for fixed $k \in \mathbb{N}$

$$\lim_{n \to \infty} |c_k - \alpha_k^{(n)}| = 0$$

or $|\alpha_k^{(n)}| \leqslant A_4$ for all $n \in \mathbb{N}$, $n \geqslant k$, with a constant $A_4 > 0$. Now by (27), (28), and (29) we get immediately for fixed $k \in \mathbb{N}$ the bound

$$\rho_n^{(k)}(f, (v), [a, b]) \leqslant A R_0^{-n}, \qquad n \in \mathbb{N},$$

$$= A \left(\frac{\sqrt{b} + \sqrt{a}}{\sqrt{b} - \sqrt{a}} \right)^{-n},$$

and the assertion is proved also for $\kappa = a$.

Remark 2. For fixed a, b the geometric factor

$$H(\kappa) := \frac{(\sqrt{b - a + \kappa} + \sqrt{\kappa})^2}{b - a} \tag{34}$$

in (16) is a strictly monotonically increasing function of κ, $0 \leqslant \kappa \leqslant a$, with $H(0) = 1$ and $H(a) = (\sqrt{b} + \sqrt{a})/(\sqrt{b} - \sqrt{a})$. For the special case $\kappa = a$ inequality (16) is given in M. Hasson [6].

By (16) for functions f holomorphic only in ellipses $E((b-a)/2 + \kappa)$, with $0 < \kappa \leqslant a$, that are ellipses not containing the "singular point" zero in the interior, the minimal deviation $\rho_n^{(k)}(f, (v), [a, b])$, $a > 0$, tends to zero geometrically with the same factor (34) as the minimal deviation $\rho_n(f, (v), [a, b])$ in approximating with all polynomials from Π_n. Moreover since

$$\rho_n(f, (v), [a, b]) \leqslant \rho_n^{(k)}(f, (v), [a, b])$$

the converse of Theorem 1 follows directly from the converse theorem of S. N. Bernstein for the approximation by all algebraic polynomials (cf. G. Meinardus [11, p. 92]).

The situation is changed if the function f is holomorphic in a region containing an ellipse $E((b-a)/2 + \kappa)$ with $\kappa > a$. The difference between our special Müntz problem and the usual approximation by polynomials in this case is elucidated by

THEOREM 2. *Let a, b, $0 < a < b$, and $k \in \mathbb{N}$ be given. Let the function f be holomorphic on the ellipse $E((b-a)/2 + \kappa)$ (cf. (15)) and the interior where κ*

satisfies $\kappa > a$. Suppose $f(z)$ is real for real z. If in addition $f^{(k)}(0) = 0$ then with a constant A

$$\rho_n^{(k)}(f, (v), [a, b]) \leqslant A \left(\frac{(\sqrt{b-a+\kappa} + \sqrt{\kappa})^2}{b-a} \right)^{-n}, \qquad n \in \mathbb{N}. \qquad (35)$$

In return for a function $f \in C[a, b]$ let inequality (35) hold with κ, $\kappa > a > 0$. Then there exists a function $\hat{f}$ holomorphic in the interior of $E((b-a)/2 + \kappa)$ whose restriction to the interval $[a, b]$ coincides with f. Moreover $\hat{f}^{(k)}(0) = 0$.

Proof. To prove the first part of the theorem we proceed just as in the proof of Theorem 1. Taking as before $\kappa_1, \kappa_2; \kappa_1 > \kappa_2 > a$ and setting again $q_1 = 1 + 2\kappa_1/(b-a)$, $R_1 = q_1 + \sqrt{q_1^2 - 1}$, $q_2 = 1 + 2\kappa_2/(b-a)$, $R_2 = q_2 + \sqrt{q_2^2 - 1}$ we obtain from (17), (18), and (21) inequality

$$\begin{aligned} |f(z) - p_n(z)| &\leqslant \sum_{v=n+1}^{\infty} |a_v| \left| T_v \left(\frac{2(z-a)}{b-a} - 1 \right) \right| \\ &\leqslant 2B(\kappa_1) \sum_{v=n+1}^{\infty} \left(\frac{R_2}{R_1} \right)^v \\ &\leqslant B_1 \left(\frac{R_2}{R_1} \right)^n \end{aligned} \qquad (36)$$

for all $z \in E((b-a)/2 + \kappa_2)$. Since $\kappa_2 > a$ the circle $K_\sigma = \{z \in \mathbb{C} : |z| < \sigma\}$, $\sigma := \kappa_2 - a$, is contained in $E((b-a)/2 + \kappa_2)$ and as in the proof of Theorem 1 we get with the Taylor expansion

$$f(z) = \sum_{v=0}^{\infty} c_v z^v$$

of f by (33) and (36) using $c_k = f^{(k)}(0)/k! = 0$ the bound

$$|\alpha_k^{(n)}| \leqslant \frac{\|f - p_n\|_{E((b-a)/2+\kappa_2)}}{\sigma^{k+1}} \leqslant B_2 \left(\frac{R_2}{R_1} \right)^n, \qquad k, n \in \mathbb{N}; k \leqslant n, k \text{ fixed}, \qquad (37)$$

for the coefficients $\alpha_k^{(n)}$ of p_n. Combining (27), (28), (29), and (37) we find

$$\rho_n^{(k)}(f, (v), [a, b]) \leqslant C_1 R_1^{-n} + B_3 \left(\frac{R_2}{R_1} \right)^n R_0^{\,n} \qquad (38)$$

with $R_0 = q_0 + \sqrt{q_0^2 - 1}$, $q_0 = 1 + 2a/(b-a)$. Now we can choose a number κ_2, $\kappa_1 > \kappa > \kappa_2 > a$, κ_2 small enough such that with

$$R = 1 + \frac{2\kappa}{b-a} + \sqrt{\left(1 + \frac{2\kappa}{b-a}\right)^2 - 1} = \frac{(\sqrt{b-a+\kappa} + \sqrt{\kappa})^2}{b-a}$$

inequality

$$\frac{R_2}{R_1} R_0^{-1} \leqslant R^{-1}$$

is satisfied. Thus (35) follows from (38). The first assertion of the converse statement is proved in the same way as the corresponding assertion for the usual approximation by polynomials without any gap (cf. G. Meinardus [11, p. 92]). With the polynomials $q_n \in \Pi_n \backslash \operatorname{span}(x^k)$ satisfying

$$\rho_n^{(k)}(f, (\nu), [a, b]) = \|f - q_n\|_{[a,b]}, \qquad n \in \mathbb{N},$$

we set $h_0 := q_0$, $h_n := q_n - q_{n-1}$; $n = 1, 2, ...$ The series

$$f(x) = \sum_{n=0}^{\infty} h_n(x)$$

is uniformly convergent in $[a, b]$. Moreover it follows as in [11] that the function

$$\hat{f}(z) = \sum_{n=0}^{\infty} h_n(z) \tag{39}$$

is uniformly convergent in any ellipse $E((b-a)/2 + \kappa_1)$ with κ_1, $0 < \kappa_1 < \kappa$, i.e., f is the restriction of the function $\hat{f}$ holomorphic in the interior of $E((b-a)/2 + \kappa)$. Remembering $h_n^{(k)}(0) = 0$, $n \in \mathbb{N}$, by applying the Weierstrass theorem to the series (39) we finally obtain

$$\hat{f}^{(k)}(0) = \sum_{n=0}^{\infty} h_n^{(k)}(0) = 0.$$

3. The Approximation of Holomorphic Functions by Müntz Polynomials

We consider first the approximation of the power function $f(x) = x^{\lambda}$, $\lambda \geqslant 0$ fixed, by help of polynomials from $\Pi_n(\lambda_\nu)$ on $[a, b]$ with $a > 0$. The following theorem (cf. Borosh, Chui, and Smith [3]) says that the minimal deviation $\rho_n(x^{\lambda}, (\lambda_\nu), [a, b])$, $a \geqslant 0$, is the smaller the closer the numbers λ_ν, $\nu = 0(1)\,n$, are to the number λ.

Theorem 3. *Let $n \in \mathbb{N}$, $\lambda \geqslant 0$, a, b, $0 \leqslant a < b$, be given. The minimal deviation*

$$\rho_n(x^{\lambda}, (\lambda_\nu), [a, b]) \tag{40}$$

with values λ_ν, $\nu = 0(1)\,n$,

$$0 \leqslant \lambda_0 < \lambda_1 < \cdots < \lambda_q < \lambda < \lambda_{q+1} < \cdots < \lambda_n, \qquad -1 \leqslant q \leqslant n,$$

is a strictly monotonically increasing function of the values $\lambda_{q+1}, \ldots, \lambda_n$ *and a strictly monotonically decreasing function of* $\lambda_0, \ldots, \lambda_q$.

Proof. The proof is just the same as the proof of Theorem 2 in Borosh, Chui, and Smith [3] where only the case $\lambda_\nu \in \mathbb{N}$, $\nu = 0(1)\,n$, has been considered.

Estimates for the minimal deviation (40) can be obtained by comparing with the values

$$\rho_n^{(k)}(x^{\gamma k}, (\gamma\nu), [a, b]) = \min_{\alpha_\nu} \left\| x^{\gamma k} - \sum_{\substack{\nu=0\\ \nu\neq k}}^{n} \alpha_\nu x^{\gamma\nu} \right\|_{[a,b]}, \qquad \gamma > 0.$$

Since

$$\left\| \sum_{\nu=0}^{n} \alpha_\nu x^{\gamma\nu} \right\|_{[a,b]} = \left\| \sum_{\nu=0}^{n} \alpha_\nu x^{\nu} \right\|_{[a^\gamma, b^\gamma]}, \tag{41}$$

we find from (12) for fixed $k \in \mathbb{N}$, $0 < \gamma < \infty$, the inequality

$$\frac{A(k)}{n^k}\left(\frac{\sqrt{b^\gamma}+\sqrt{a^\gamma}}{\sqrt{b^\gamma}-\sqrt{a^\gamma}}\right)^{-n} \leqslant \rho_n^{(k)}(x^{\gamma k}, (\gamma\nu), [a, b]) \leqslant \frac{B(k)}{n^k}\left(\frac{\sqrt{b^\gamma}+\sqrt{a^\gamma}}{\sqrt{b^\gamma}-\sqrt{a^\gamma}}\right)^{-n}, \tag{42}$$

$n \in \mathbb{N}$, with constants $A(k)$, $B(k)$ not depending on n. We notice that given a, b, $0 < a < b$, the factor

$$g(\gamma) := \frac{\sqrt{b^\gamma}+\sqrt{a^\gamma}}{\sqrt{b^\gamma}-\sqrt{a^\gamma}}$$

is strictly monotonically decreasing in γ, $\gamma > 0$. This follows from

$$g'(\gamma) = \frac{\sqrt{b^\gamma}\,\sqrt{a^\gamma}(\log a - \log b)}{(\sqrt{b^\gamma}-\sqrt{a^\gamma})^2} < 0.$$

More generally we get from (42) by aid of Theorem 3 the following

THEOREM 4. *Let* $k \in \mathbb{N}$, a, b, $0 < a < b$, *be given. Suppose the sequence* (λ_ν) (*cf.* (2)) *satisfies* $0 \leqslant d \leqslant \lambda_{\nu+1} - \lambda_\nu \leqslant D < \infty$, $\nu \in \mathbb{N}$. *Then for the minimal deviation*

$$\rho_n^{(k)}(x^{\lambda_k}, (\lambda_\nu), [a, b]) := \min_{\alpha_\nu} \left\{ \left\| x^{\lambda_k} - \sum_{\substack{\nu=0\\ \nu\neq k}}^{n} \alpha_\nu x^{\lambda_\nu} \right\|_{[a,b]} \right\}$$

the bound

$$\frac{A}{n^k}\left(\frac{\sqrt{b^d}+\sqrt{a^d}}{\sqrt{b^d}-\sqrt{a^d}}\right)^{-n} \leqslant \rho_n^{(k)}(x^{\lambda_k}, (\lambda_v), [a, b]) \leqslant \frac{B}{n^r}\left(\frac{\sqrt{b^D}+\sqrt{a^D}}{\sqrt{b^D}-\sqrt{a^D}}\right)^{-n} \tag{43}$$

holds for all $n \in \mathbb{N}$ *with* $r := [\lambda_k/D]$ *and constants* A, B *not depending on* n. (*Here* $[x]$ *for* $x \in \mathbb{R}$ *denotes the largest natural number* k, $k \leqslant x$.)

Proof. Setting with k, $k \leqslant n$,

$$r := [\lambda_k/D], \qquad \alpha := \lambda_k - Dr \geqslant 0$$

and using

$$\lambda_{k+\mu} \leqslant \lambda_k + D\mu, \qquad \mu = 1(1)\, n-k,$$
$$\lambda_{k-\mu} \geqslant \lambda_k - D\mu, \qquad \mu = 1(1)\, r,$$

we deduce from Theorem 3 that

$$\begin{aligned}\rho_n^{(k)}(x^{\lambda_k}, (\lambda_v), [a, b]) &\leqslant \rho_{n+r-k}^{(r)}(x^{\lambda_k}, (\lambda_k + D(v-r)), [a, b]) \\ &= \rho_{n+r-k}^{(r)}(x^{rD+\alpha}, (vD+\alpha), [a, b]).\end{aligned} \tag{44}$$

By $\rho_{n+r-k}^{(r)}(x^{\lambda_k}, (\lambda_k + D(v-r)), [a, b])$ we have denoted the minimal deviation in approximating $f(x) = x^{\lambda_k}$ by the functions $x^{\lambda_k + D(v-r)}$, $v = 0(1)\, n+r-k$, $v \neq r$. Since

$$\left\| \sum_{v=0}^{n} \alpha_v x^{\lambda_v + \alpha} \right\|_{[a,b]} \leqslant b^\alpha \left\| \sum_{v=0}^{n} \alpha_v x^{\lambda_v} \right\|_{[a,b]}$$

it follows with (42) and (44) that

$$\begin{aligned}\rho_n^{(k)}(x^{\lambda_k}, (\lambda_v), [a, b]) &\leqslant b^\alpha \rho_{n+r-k}^{(r)}(x^{rD}, (vD), [a, b]) \\ &\leqslant b^\alpha B(r) \frac{1}{(n+r-k)^r}\left(\frac{\sqrt{b^D}+\sqrt{a^D}}{\sqrt{b^D}-\sqrt{a^D}}\right)^{-n-r+k}\end{aligned}$$

Taking a constant B satisfying

$$B \geqslant b^\alpha B(r)\left(\frac{n}{n+r-k}\right)^r \left(\frac{\sqrt{b^D}+\sqrt{a^D}}{\sqrt{b^D}-\sqrt{a^D}}\right)^{-r+k}, \qquad n \in \mathbb{N},$$

the bound

$$\rho_n^{(k)}(x^{\lambda_k}, (\lambda_v), [a, b]) \leqslant \frac{B}{n^r}\left(\frac{\sqrt{b^D}+\sqrt{a^D}}{\sqrt{b^D}-\sqrt{a^D}}\right)^{-n}, \qquad n \in \mathbb{N},$$

is established. To prove the lower bound in (43) we assume $d>0$ since for $d=0$ the bound is trivially satisfied. Putting $\beta := \lambda_k - kd$ we get by Theorem 3

$$\rho_n^{(k)}(x^{\lambda_k}, (\lambda_\nu), [a, b]) \geqslant \rho_n^{(k)}(x^{\lambda_k}, (\lambda_k + d(\nu - k)), [a, b])$$
$$= \rho_n^{(k)}(x^{kd+\beta}, (d\nu + \beta), [a, b])$$

and in view of

$$\left\| \sum_{\nu=0}^{n} \alpha_\nu x^{\lambda_\nu + \beta} \right\|_{[a,b]} \geqslant a^\beta \left\| \sum_{\nu=0}^{n} \alpha_\nu x^{\lambda_\nu} \right\|_{[a,b]}$$

the bound (42) leads to the left inequality of (43),

$$\rho_n^{(k)}(x^{\lambda_k}, (\lambda_\nu), [a, b]) \geqslant a^\beta A(k) \frac{1}{n^k} \left(\frac{\sqrt{b^d} + \sqrt{a^d}}{\sqrt{b^d} - \sqrt{a^d}} \right)^{-n}$$
$$\geqslant A \frac{1}{n^k} \left(\frac{\sqrt{b^d} + \sqrt{a^d}}{\sqrt{b^d} - \sqrt{a^d}} \right)^{-n}, \qquad n \in \mathbb{N},$$

and the theorem is proved.

Given a sequence (λ_ν), $0 \leqslant d \leqslant \lambda_{\nu+1} - \lambda_\nu \leqslant D < \infty$, $\nu \in \mathbb{N}$, and a number $\lambda \geqslant 0$ with

$$\min_{\nu \in \mathbb{N}} |\lambda - \lambda_\nu| =: \hat{\delta} > 0, \qquad \lambda_{k-1} < \lambda < \lambda_k, k \in \mathbb{N},$$

then by Theorem 4 for the minimal deviation $\rho_n(x^\lambda, (\lambda_\nu), [a, b])$ in approximating the function $f(x) = x^\lambda$ by polynomials from $\Pi_n(\lambda_\nu)$ on $[a, b]$, $a > 0$, inequality

$$\frac{A}{n^k} \left(\frac{\sqrt{b^\delta} + \sqrt{a^\delta}}{\sqrt{b^\delta} - \sqrt{a^\delta}} \right)^{-n} \leqslant \rho_n(x^\lambda, (\lambda_\nu), [a, b]) \leqslant \frac{B}{n^r} \left(\frac{\sqrt{b^D} + \sqrt{a^D}}{\sqrt{b^D} - \sqrt{a^D}} \right)^{-n}, \qquad n \in \mathbb{N}, \tag{45}$$

holds with $r := [\lambda/D]$ and $\delta := \min(d, \hat{\delta})$. In [1] the upper bound

$$\rho_n(x^\lambda, (\lambda_\nu), [a, b]) \leqslant C(\lambda) \sqrt{n} \left(\frac{\sqrt{b^D}}{\sqrt{b^D} - \sqrt{a^D}} \right)^{-n}, \qquad n \in \mathbb{N},$$

has been given.

But also more generally for the approximation on intervals $[a, b]$ with $a > 0$ (in contrast to the approximation on $[0, b]$) the quantity $\rho_n(f, (\lambda_\nu), [a, b])$ has a geometric decrease for all functions holomorphic in a sufficiently large region around the approximation interval $[a, b]$. This is stated in

THEOREM 5. *Let $a, b, 0 < a < b$, be given and let the sequence (λ_ν) satisfy $0 < d \leqslant \lambda_{\nu+1} - \lambda_\nu \leqslant D < \infty$, $\nu \in \mathbb{N}$. Suppose the function f is holomorphic in the interior of K_R and continuous on K_R, $K_R := \{z \in \mathbb{C}: |z| \leqslant R\}$ where R is a number $R > b$. Further let $f(z)$ be real for real z. Then for any q,*

$$0 < q < \sigma := \min\left\{\left(\frac{R}{b}\right)^d, \frac{\sqrt{b^D} + \sqrt{a^D}}{\sqrt{b^D} - \sqrt{a^D}}\right\}, \tag{46}$$

there exists a constant $A = A(q)$ not depending on n such that

$$\rho_n(f, (\lambda_\nu), [a, b]) \leqslant A q^{-n} \tag{47}$$

holds for all $n \in \mathbb{N}$.

Proof. For the coefficients c_ν, $\nu \in \mathbb{N}$, of the Taylor expansion

$$f(z) = \sum_{\nu=0}^{\infty} c_\nu z^\nu$$

of f we get by Cauchy's integral formula the bound

$$\begin{aligned} |c_\nu| &= \left|\frac{1}{2\pi i}\int_{|\xi|=R} \frac{f(\xi)}{\xi^{\nu+1}}\, d\xi\right|, \\ &\leqslant \frac{M}{R^\nu}, \end{aligned} \qquad \nu \in \mathbb{N}, \tag{48}$$

with $M := \max_{|z|=R}\{|f(z)|\}$. For any $m \in \mathbb{N}$ inequality

$$\begin{aligned} \rho_{[m/d]}(f, (\lambda_\nu), [a, b]) \leqslant \sum_{\lambda=0}^{m} |c_\lambda|\, \rho_{[m/d]}(x^\lambda, (\lambda_\nu), [a, b]) \\ + \sum_{\lambda=m+1}^{\infty} |c_\lambda|\, b^\lambda \end{aligned} \tag{49}$$

is valid. We seek for upper bounds for the minimal deviations

$$\rho_{[m/d]}(x^\lambda, (\lambda_\nu), [a, b]), \qquad \lambda \in \mathbb{N}, \lambda \leqslant m. \tag{50}$$

If $\lambda \geqslant \lambda_0$ then the value (50) is less or equal to the minimal deviation in approximating $g(x) = x^\lambda$ by the power functions $x^{\lambda+D}, \ldots, x^{\lambda+D([m/d]-[\lambda/d])}$.

This follows from Theorem 3 since $\lambda_k \leqslant \lambda \leqslant \lambda_{k+1}$, $k \in \mathbb{N}$, implies $\lambda \geqslant kd$ and $[\lambda/d] \geqslant k$. Hence using (41) we find

$$\begin{aligned} \rho_{[m/d]}(x^\lambda, (\lambda_v), [a, b]) &\leqslant \rho^{(0)}_{[m/d] - [\lambda/d]}(x^\lambda, (\lambda + vD), [a, b]) \\ &\leqslant b^\lambda \rho^{(0)}_{[m/d] - [\lambda/d]}(1, (vD), [a, b]) \\ &= b^\lambda \rho^{(0)}_{[m/d] - [\lambda/d]}(1, (v), [a^D, b^D]) \end{aligned} \tag{51}$$

for $\lambda_0 \leqslant \lambda \leqslant m$. Now by (10)

$$\rho_n^{(0)}(1, (v), [\alpha, \beta]) = \frac{1}{\left| T_n\left(1 + \dfrac{2\alpha}{\beta - \alpha}\right)\right|}$$

for any α, β, $0 \leqslant \alpha < \beta$. Using (13) we obtain with $x = 1 + 2\alpha/(\beta - \alpha)$, $v = x + \sqrt{x^2 - 1}$ the bound

$$T_n\left(1 + \frac{2}{\beta - \alpha}\right) = \frac{1}{2}\left(v^n + \frac{1}{v^n}\right) > \frac{1}{2} v^n = \frac{1}{2}\left(\frac{\sqrt{\beta} + \sqrt{\alpha}}{\sqrt{\beta} - \sqrt{\alpha}}\right)^n, \qquad n \in \mathbb{N},$$

and in view of (51) we get

$$\rho_{[m/d]}(x^\lambda, (\lambda_v), [a, b]) \leqslant 2b^\lambda \left(\frac{\sqrt{b^D} + \sqrt{a^D}}{\sqrt{b^D} - \sqrt{a^D}}\right)^{[m/d] + [\lambda/d]}, \qquad n \in \mathbb{N}, \tag{52}$$

for λ, $\lambda_0 \leqslant \lambda \leqslant m$. For the finitely many $\lambda \in \mathbb{N}$, $\lambda \leqslant \lambda_0$, Theorem 3 yields by applying

$$\left\| x^\lambda - \sum_{v=0}^{m} \alpha_v x^{\lambda_0 + vD} \right\|_{[a,b]} \leqslant b^{\lambda_0} \left\| x^{(\lambda - \lambda_0)/D} - \sum_{v=0}^{m} \alpha_v x^v \right\|_{[a^D, b^D]}$$

inequality

$$\begin{aligned} \rho_{[m/d]}(x^\lambda, (\lambda_v), [a, b]) &\leqslant \rho_{[m/d]}(x^\lambda, (\lambda_0 + vD), [a, b]) \\ &\leqslant b^{\lambda_0} \rho_{[m/d]}(x^{(\lambda - \lambda_0)/D}, (v), [a^D, b^D]). \end{aligned} \tag{53}$$

The function $h(x) = x^{(\lambda - \lambda_0)/D}$ is the restriction of a function holomorphic in any ellipse around the interval $[a, b]$ provided that the ellipse does not contain the zero point. Taking account of the transformation of $[a^D, b^D]$ onto $[-1, 1]$ a result of S. N. Bernstein (cf. [2]) says that inequality

$$\rho_n(x^{(\lambda - \lambda_0)/D}, (v), [a^D, b^D]) \leqslant B \cdot \kappa^{-n} \cdot n^{(\lambda_0 - \lambda - D)/D}, \qquad 1 \leqslant n \in \mathbb{N},$$

is valid with $\kappa = (\sqrt{b^D} + \sqrt{a^D})/(\sqrt{b^D} - \sqrt{a^D})$ and a constant B not depending on n. Hence by (53) there exists a constant C such that for $m \in \mathbb{N}$, $d \leqslant m$,

$$\rho_{\lceil m/d \rceil}(x^{\lambda}, (v), [a, b]) \leqslant C b^{\lambda_0} \left(\frac{\sqrt{b^D} + \sqrt{a^D}}{\sqrt{b^D} - \sqrt{a^D}} \right)^{-\lceil m/d \rceil} \cdot \left[\frac{m}{d} \right]^{\lambda_0/D} \tag{54}$$

for all $\lambda \in \mathbb{N}$, $\lambda < \lambda_0$. Combining (48), (49), (52), and (54) we obtain

$$\begin{aligned} &\rho_{\lceil m/d \rceil}(f, (\lambda_v), [a, b]) \\ &\leqslant C M b^{\lambda_0} \sum_{\lambda=0}^{\lceil \lambda_0 \rceil} R^{\lambda} \left[\frac{m}{d} \right]^{\lambda_0/D} \left(\frac{\sqrt{b^D} + \sqrt{a^D}}{\sqrt{b^D} - \sqrt{a^D}} \right)^{\lceil m/d \rceil} \\ &\quad + 2M \sum_{\lambda = \lceil \lambda_0 \rceil + 1}^{m} \left(\left(\frac{R}{b} \right)^{d} \right)^{\lambda/d} \left(\frac{\sqrt{b^D} + \sqrt{a^D}}{\sqrt{b^D} - \sqrt{a^D}} \right)^{\lceil \lambda/d \rceil - \lceil m/d \rceil} + M \sum_{\lambda = m+1}^{\infty} \left(\frac{R}{b} \right)^{\lambda} \\ &\leqslant k_1 \left(\frac{\sqrt{b^D} + \sqrt{a^D}}{\sqrt{b^D} - \sqrt{a^D}} \right)^{-\lceil m/d \rceil} \cdot \left[\frac{m}{d} \right]^{\lambda_0/D} + k_2 m \cdot \sigma^{-\lceil m/d \rceil} + k_3 \cdot \left(\left(\frac{R}{b} \right)^{d} \right)^{-\lceil m/d \rceil} \end{aligned}$$

with

$$\sigma := \min \left\{ \left(\frac{R}{b} \right)^{d}, \frac{\sqrt{b^D} + \sqrt{a^D}}{\sqrt{b^D} - \sqrt{a^D}} \right\}$$

and constants k_1, k_2, k_3 not depending on $m \in \mathbb{N}$. Thus to any number $q < \sigma$ there exists a constant $A = A(q)$ such that (47) holds as stated.

The upper bound (47) for the minimal deviation $\rho_n(f, (\lambda_v), [a, b])$ depends on the value σ (cf. (46)). For functions f holomorphic in a sufficiently large region, i.e.,

$$\left(\frac{R}{b} \right)^{d} \geqslant \frac{\sqrt{b^D} + \sqrt{a^D}}{\sqrt{b^D} - \sqrt{a^D}}, \tag{55}$$

the geometric factor σ is bounded (from above) by the right side of (55) which doesn't depend on f and the rate of convergence increases with the distance of the interval $[a, b]$ from the origin.

4. A Converse Theorem

Regarding again inequality (47) with q given in (46) the bound in Theorem 5 suggests that for fixed interval $[a, b]$, $a > 0$, generally even for entire functions f the geometric rate of $\rho_n(f, (\lambda_v), [a, b])$ is bounded by a number depending only on the distance of the interval $[a, b]$ from the zero point and on the difference $b-a$. More precisely the following generalisation of the converse result of Theorem 2 holds.

THEOREM 6. *Let $a, b, 0<a<b$, be given and let the sequence (λ_ν) satisfy $0<d\leqslant\lambda_{\nu+1}-\lambda_\nu\leqslant D<\infty$, $\nu\in\mathbb{N}$. Suppose, for a function $f\in C[a,b]$, the inequality*

$$\rho_n(f, (\lambda_\nu), [a,b])\leqslant Ar^{-n}, \qquad n\in\mathbb{N},$$

holds with constants A and r, where

$$r>\kappa:=\frac{2eb^D}{a^D}\cdot\frac{\sqrt{b^D}+\sqrt{a^D}}{\sqrt{b^D}-\sqrt{a^D}}. \tag{56}$$

Then there exists a Müntz series

$$\hat{f}(z)=\sum_{\nu=0}^{\infty} c_\nu z^{\lambda_\nu},$$

absolutely convergent in $\mathring{K}_R:=\{z\in\mathbb{C}_{\log}: |z|<R\}$, with $R=b\cdot(r/\kappa)^{1/D}$, whose restriction to the interval $[a,b]$ coincides with the given function f.

Proof. For convenience we apply a transformation to the approximation problem. Setting

$$\tilde{x}:=x/b, \qquad F(\tilde{x})=f(b\tilde{x})=f(x) \tag{57}$$

by assumption equivalently to

$$\left\|f(x)-\sum_{\nu=0}^{n}\alpha_\nu^{(n)}x^{\lambda_\nu}\right\|_{[a,b]}\leqslant Ar^{-n}, \qquad n\in\mathbb{N},$$

inequality

$$\left\|F(\tilde{x})-\sum_{\nu=0}^{n}\alpha_\nu^{(n)}b^{\lambda_\nu}\tilde{x}^{\lambda_\nu}\right\|_{[a/b,1]}\leqslant Ar^{-n}, \qquad n\in\mathbb{N},$$

holds. Putting $\alpha:=a/b$, $0<\alpha<1$, this means that

$$\rho_n(F, (\lambda_\nu), [\alpha, 1])\leqslant Ar^{-n}, \qquad n\in\mathbb{N}.$$

With polynomials $q_n\in\Pi_n(\lambda_\nu)$ satisfying $\|F-q_n\|_{[\alpha,1]}\leqslant Ar^{-n}$, $n\in\mathbb{N}$, we define $p_n:=q_n-q_{n-1}$; $n=1,2,\ldots$; $p_0:=q_0$. Then the series

$$F(\tilde{x})=q_0(\tilde{x})+\sum_{n=1}^{\infty}(q_n(\tilde{x})-q_{n-1}(\tilde{x}))=\sum_{n=0}^{\infty}p_n(\tilde{x})$$

is uniformly convergent in $[\alpha, 1]$ and moreover it follows

$$\begin{aligned} \|p_n\|_{[\alpha,1]} &\leqslant \|F-q_n\|_{[\alpha,1]} + \|F-q_{n-1}\|_{[\alpha,1]}, \qquad n\in\mathbb{N}, \\ &\leqslant Br^{-n}, \end{aligned} \tag{58}$$

with a constant B. For the polynomials

$$p_n(\tilde{z}) := \sum_{\nu=0}^{n} a_\nu^{(n)} \tilde{z}^{\lambda_\nu} \in \Pi_n(\lambda_\nu) \tag{59}$$

clearly

$$|p_n(\tilde{z})| \leqslant \sum_{\nu=0}^{n} |a_\nu^{(n)}|\, |\tilde{z}|^{\lambda_\nu} \tag{60}$$

is correct where $\tilde{z}$ is an element of the Riemann surface of the logarithm $\mathbb{C}_{\log}$. Since

$$\rho_n^{(k)}(\tilde{x}^{\lambda_k}, (\lambda_\nu), [\alpha, 1]) \leqslant \frac{\|p_n\|_{[\alpha,1]}}{|a_k^{(n)}|}$$

we have

$$|a_k^{(n)}| \leqslant \frac{\|p_n\|_{[\alpha,1]}}{\rho_n^{(k)}(\tilde{x}^{\lambda_k}, (\lambda_\nu), [\alpha, 1])} \tag{61}$$

for all $k, n\in\mathbb{N}$, $k\leqslant n$. By Theorem 3 the value $\rho_n^{(k)}(\tilde{x}^{\lambda_k}, (\lambda_\nu), [\alpha, 1])$ is not smaller than the minimal deviation in approximating $g(\tilde{x}) = \tilde{x}^{\lambda_k}$ by the functions $\tilde{x}^{\lambda_k+(\nu-k)d}$, $\nu = 0(1)\,n$, $\nu\neq k$. Consequently

$$\begin{aligned} \rho_n^{(k)}(\tilde{x}^{\lambda_k}, (\lambda_\nu), [\alpha, 1]) &\geqslant \rho_n^{(k)}(\tilde{x}^{\lambda_k}, (\lambda_k + (\nu-k)\,d), [\alpha, 1]) \\ &\geqslant \alpha^{\lambda_k - kd}\rho_n^{(k)}(\tilde{x}^{dk}, (\nu d), [\alpha, 1]) \\ &= \alpha^{\lambda_k - kd}\rho_n^{(k)}(\tilde{x}^{k}, (\nu), [\alpha^d, 1]). \end{aligned} \tag{62}$$

We remember (10),

$$\rho_n^{(k)}(\tilde{x}^k, (\nu), [\alpha^d, 1]) = \frac{k!\,(1-\alpha^d)^k}{2^k\left|T_n^{(k)}\left(1+\dfrac{2\alpha^d}{1-\alpha^d}\right)\right|}, \tag{63}$$

for $k, n \in \mathbb{N}, k \leqslant n$. Applying the inequality of W. A. Markoff (cf. [11]) to T_n on the interval $[-q, q], q > 1$, we find the bound

$$|T_n^{(k)}(q)| \leqslant \frac{2^{2k} k!}{(2q)^k} \frac{n}{n+k} \binom{n+k}{n-k} \|T_n\|_{[-q,q]}. \tag{64}$$

From (13) it follows with $q > 1$

$$\|T_n\|_{[-q,q]} = T_n(q) \leqslant (q + \sqrt{q^2 - 1})^n \tag{65}$$

for all $n \in \mathbb{N}$. Combining

$$\frac{n}{n+k} \binom{n+k}{n-k} = \frac{\prod_{\nu=0}^{k-1} (n^2 - \nu^2)}{(2k)!} \leqslant \frac{n^{2k}}{(2k)!} \leqslant \frac{n^n}{n!} < e^n \qquad \text{for } k \leqslant n$$

with (63), (64), and (65) we get

$$\rho_n^{(k)}(\tilde{x}^k, (\nu), [\alpha^d, 1]) \geqslant \frac{(1-\alpha^d)^k q^k}{2^{2k} e^n (q + \sqrt{q^2 - 1})^n}$$

for all $k, n \in \mathbb{N}, k \leqslant n$ with $q = 1 + 2\alpha^d/(1 - \alpha^d)$. Since

$$q^k (1 - \alpha^d)^k = (1 + \alpha^d)^k \qquad \text{and} \qquad q + \sqrt{q^2 - 1} = \frac{1 + \sqrt{\alpha^d}}{1 - \sqrt{\alpha^d}}$$

we find

$$\rho_n^{(k)}(\tilde{x}^k, (\nu), [\alpha^d, 1]) \geqslant \frac{(1+\alpha^d)^k}{2^{2k} e^n} \left(\frac{1 + \sqrt{\alpha^d}}{1 - \sqrt{\alpha^d}} \right)^{-n}$$

and in view of (62) using $0 < \alpha < 1$ finally

$$\begin{aligned} \rho_n^{(k)}(\tilde{x}^{\lambda_k}, (\lambda_\nu), [\alpha, 1]) &\geqslant \frac{\alpha^{\lambda_k - kd} (2\alpha^d)^k}{2^{2k} e^n} \left(\frac{1 + \sqrt{\alpha^d}}{1 - \sqrt{\alpha^d}} \right)^{-n} \\ &\geqslant \frac{\alpha^{Dn}}{2^n e^n} \left(\frac{1 + \sqrt{\alpha^d}}{1 - \sqrt{\alpha^d}} \right)^{-n} \end{aligned}$$

for all $k, n \in \mathbb{N}, k \leqslant n$. Hence by (58) and (61) with

$$\kappa = \frac{2e}{\alpha^D} \frac{1 + \sqrt{\alpha^d}}{1 - \sqrt{\alpha^d}} \qquad \text{where } \alpha = \frac{a}{b}$$

(cf. (56)) inequality

$$|a_k^{(n)}| \leqslant B \left(\frac{\kappa}{r} \right)^n$$

is valid for the coefficients $a_k^{(n)}$, $k = 0(1)\,n$, of p_n, $n \in \mathbb{N}$ (cf. (59)). Thus (60) implies

$$|p_n(\tilde{z})| \leqslant \sum_{k=0}^{n} |a_k^{(n)}|\, |\tilde{z}|^{\lambda_k} \leqslant B \sum_{k=0}^{n} \left(\frac{\kappa}{r}\right)^n \left(\frac{r}{\kappa}\right)^n (1-\varepsilon)^n, \qquad n \in \mathbb{N}, \quad (66)$$
$$\leqslant B(n+1)(1-\varepsilon)^n,$$

for all $\tilde{z} \in \mathbb{C}_{\log}$, $|\tilde{z}| \leqslant ((r/\kappa)(1-\varepsilon))^{1/D}$ where ε is any number $\varepsilon > 0$ satisfying $(r/\kappa)(1-\varepsilon) \geqslant 1$. From

$$|\hat{F}(\tilde{z})| \leqslant \sum_{n=0}^{\infty} |p_n(\tilde{z})| \leqslant B \sum_{n=0}^{\infty} (n+1)(1-\varepsilon)^n < \infty$$

we deduce that the series

$$\hat{F}(\tilde{z}) = \sum_{n=0}^{\infty} p_n(\tilde{z}) \qquad (67)$$

is uniformly convergent for all $\tilde{z} \in \mathbb{C}_{\log}$, $|\tilde{z}| \leqslant ((r/\kappa)(1-\varepsilon))^{1/D}$. Moreover expansion (67) of $\hat{F}$ as an absolutely convergent double sum allows a change of summation which leads to a representation

$$\hat{F}(\tilde{z}) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k^{(n)} \tilde{z}^{\lambda_k} \right) = \sum_{k=0}^{\infty} \left(\sum_{n=0}^{\infty} a_k^{(n)} \right) \tilde{z}^{\lambda_k}$$
$$= \sum_{\nu=0}^{\infty} \tilde{c}_\nu \tilde{z}^{\lambda_\nu} \qquad (68)$$

absolutely convergent for $\tilde{z} \in \mathbb{C}_{\log}$, $|\tilde{z}| \leqslant ((r/\kappa)(1-\varepsilon))^{1/D}$. Since $\varepsilon > 0$ may be chosen arbitrarily small the series (68) is absolutely convergent in $|\tilde{z}| < (r/\kappa)^{1/D}$. Taking account of transformation (57) by setting

$$z = b\tilde{z}, \qquad \hat{f}(z) = \hat{F}(\tilde{z})$$

we have shown that the series

$$\hat{f}(z) = \sum_{\nu=0}^{\infty} \frac{\tilde{c}_\nu}{b^{\lambda_\nu}} z^{\lambda_\nu} = \sum_{\nu=0}^{\infty} c_\nu z^{\lambda_\nu}$$

is absolutely convergent for all $z \in \mathbb{C}_{\log}$, $|z| < b \cdot (r/\kappa)^{1/D}$. Furthermore the restriction of the function $\hat{f}$ to the interval $[a, b]$ is just the given function $f(x) = F(x/b)$.

We notice that the estimates in the proofs of the preceding Theorem 5 and Theorem 6 have led to assertions which describe the situation qualitatively. They don't give sharp results in general.

5. Linear Approximation by Exponential Sums

In Section 1 by help of transformation (4) it has been shown that "Müntz problems" are equivalent to corresponding problems of approximation by exponential sums. Now we summarize the results above in terms of the (linear) approximation by exponential sums.

By transformation (4) intervals $[0, b]$ are mapped onto infinite intervals $[\alpha, \infty]$ with $\alpha = -\log b$ and intervals $[a, b]$, $a > 0$, correspond to finite intervals $[\alpha, \beta]$, $\alpha = -\log b$, $\beta = -\log a$. Moreover taking the continuation of (4) to complex variables,

$$z = e^{-s}, \qquad s = t + i\tau,$$

$$s = -\log z, \qquad z \in \mathbb{C}_{\log},$$

we see that a circle $K_R := \{z \in \mathbb{C}_{\log}: |z| \leqslant R\}$, $R > 0$, corresponds to a right half plane $H_R := \{s \in \mathbb{C}: \operatorname{re} s \geqslant -\log R\}$. Hence the results in [14] suggest that a geometric rate of the minimal deviation $\delta_n(f, (\lambda_\nu), [\alpha, \infty])$ (cf. (6)) in approximating a function F by exponential sums from $\Delta_n(\lambda_\nu)$ (cf. (5)) where the fixed sequence (λ_ν) satisfies $0 < d \leqslant \lambda_{\nu+1} - \lambda_\nu$, $\nu \in \mathbb{N}$, occurs exactly for those functions F which are restrictions of Dirichlet series

$$\hat{F}(s) = \sum_{\nu=0}^{\infty} c_\nu e^{-\lambda_\nu s},$$

convergent in certain right half planes H_R. Whereas for the approximation on finite intervals $[\alpha, \beta]$ the minimal deviation $\delta_n(F, (\lambda_\nu), [\alpha, \beta])$ for the exponential approximation where the numbers λ_ν, $\nu \in \mathbb{N}$, satisfy $\lambda_{\nu+1} - \lambda_\nu \leqslant D < \infty$ tends to zero geometrically for all functions F holomorphic in certain regions around the interval $[\alpha, \beta]$. We give an

Example. Let us consider the approximation of the function $F(t) = 1/(1+t)$ by exponential sums of the form ($\lambda_\nu = \alpha\nu$, $\nu \in \mathbb{N}$)

$$d_n(t) = \sum_{\nu=0}^{n} a_\nu e^{-\alpha\nu t} \tag{69}$$

on the interval $[0, \infty]$ resp. $[0, 1]$, where α is a fixed number $\alpha > 0$. This function F is holomorphic in the half plane $\operatorname{re} s > -1$ but not representable as a Dirichlet series. We choose $\alpha = \frac{1}{2}$. The computed minimal deviations $\delta_n[0, \infty] := \delta_n(F, (\nu/2), [0, \infty])$ resp. $\delta_n[0, 1] := \delta_n(F, (\nu/2), [0, 1])$ are

TABLE I

n	$\delta_n[0, \infty]$	$\delta_n[0, 1]$	$\rho_n[0, 1]$
2	$4.5 \cdot 10^{-2}$	$3.3 \cdot 10^{-3}$	$7.3 \cdot 10^{-3}$
3	$3.4 \cdot 10^{-2}$	$3.9 \cdot 10^{-4}$	$1.3 \cdot 10^{-3}$
4	$3.1 \cdot 10^{-2}$	$4.7 \cdot 10^{-5}$	$2.2 \cdot 10^{-4}$
5	$2.8 \cdot 10^{-2}$	$5.4 \cdot 10^{-6}$	$3.7 \cdot 10^{-5}$
6	$2.6 \cdot 10^{-2}$	$6.6 \cdot 10^{-7}$	$6.4 \cdot 10^{-6}$

listed in Table I for $n = 2, \ldots, 6$. The last column of Table I gives as comparison the minimal deviations $\rho_n[0, 1] := \rho_n(F, (v), [0, 1])$ for the approximation of $F(x) = 1/(1 + x)$ by algebraic polynomials on the interval $[0, 1]$.

Table II contains the ratios of consecutive minimal deviations.

The ratios for the exponential approximation on $[0, \infty]$ (first column, Table II) indicate that a geometric convergence of $\delta_n[0, \infty]$ cannot be expected. For the approximation on $[0, 1]$ a geometric rate occurs in the exponential case as in the polynomial case. In fact the theorem of S. N. Bernstein (cf. [11]) leads to the asymptotic relation

$$\limsup_{n \to \infty} (\rho_n[0, 1])^{1/n} = \frac{1}{3 + \sqrt{10}}, \qquad 3 + \sqrt{10} \approx 6.16,$$

and with some transformation arguments we find

$$\limsup_{n \to \infty} (\delta_n[0, 1])^{1/n} = \frac{1}{\kappa} \qquad \text{with } k = \frac{1 + \sqrt{e} + \sqrt{e + 2\sqrt{e}}}{\sqrt{e} - 1} \approx 7.86.$$

The optimal choice of α for the approximation of $F(t) = 1/(1 + t)$ on $[0, 1]$ by sums of the form (69) is the number $\alpha_0 = \log((1 + \sqrt{5})/2) \approx 0.4812$ with an asymptotic rate

$$\limsup_{n \to \infty} (\delta_n(F, (\alpha_0 v), [0, 1]))^{1/n} = \frac{1}{\kappa_0}, \qquad \kappa_0 = \frac{\sqrt{(1 + \sqrt{5})/2} + 1}{\sqrt{(1 + \sqrt{5})/2} - 1} \approx 8.35.$$

TABLE II

n	$\dfrac{\delta_n[0, \infty]}{\delta_{n+1}[0, \infty]}$	$\dfrac{\delta_n[0, 1]}{\delta_{n+1}[0, 1]}$	$\dfrac{\rho_n[0, 1]}{\rho_{n+1}[0, 1]}$
2	1.30	8.46	5.62
3	1.11	8.30	5.91
4	1.10	8.28	5.95
5	1.08	8.18	5.96

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Printed in Belgium